Recall: $K/\mathbb{Q}$ finite, $n := [K:\mathbb{Q}]$ (1)

$$\zeta_K(s) = \varkappa \cdot \frac{1}{s-1} + \underbrace{g(s)}_{\text{hol. on } \operatorname{Re} s > 1-\frac{1}{n}}$$

$$\varkappa = \frac{2^{r_1}(2\pi)^{r_2} \cdot R_K \cdot h_K}{w \cdot \sqrt{|\Delta_K|}}$$

Next aim: Get h_K from $\zeta_K(s)$

Want: Find a way to evaluate $\lim\limits_{s \searrow 1} (s-1) \cdot \zeta_K(s) = \varkappa$

Better: Evaluate

$$\lim_{s \searrow 1} \frac{\zeta_K(s)}{\zeta(s)} = \varkappa$$

Example: $K/\mathbb{Q}$ quadratic

Set $S_{ram} := \{ p>0 \text{ prime} \mid p \text{ ramified in } \mathcal{O}_K \}$

$S_{inert} := \{ \ldots\ldots \mid p \text{ inert in } \mathcal{O}_K \}$

$S_{split} := \{ \ldots\ldots \mid p \text{ split in } \mathcal{O}_K \}$

$$\Rightarrow \quad \zeta_K(s) = \prod_{\mathfrak{p} \trianglelefteq \mathcal{O}_K} \frac{1}{1 - N\mathfrak{p}^{-s}} \qquad \overbrace{\left(\frac{1}{1-p^{-s}}\right)\cdot\left(\frac{1}{1+p^{-s}}\right)}^{\frac{1}{1-p^{-2s}}}$$

$$= \prod_{p \in S_{ram}} \frac{1}{1-p^{-s}} \cdot \prod_{p \in S_{inert}} \frac{1}{1-p^{-2s}}$$

$$\cdot \prod_{p \in S_{split}} \left(\frac{1}{1-p^{-s}}\right)^2$$

$$= \zeta(s)\cdot \underbrace{\prod_{p\in S_{inert}} \frac{1}{1+p^{-s}} \cdot \prod_{p\in S_{split}} \frac{1}{1-p^{-s}}}_{L(\chi_K,\, s)}$$

$$\begin{tikzcd}
\chi_K : (\mathbb{Z}/\Delta_K)^\times \arrow[r, two heads] & \{\pm 1\} \subseteq \mathbb{C}^\times \\
\mathrm{Gal}(\mathbb{Q}(\zeta_{|\Delta_K|})/\mathbb{Q}) \arrow[u, "\cong"] \arrow[r, two heads] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Gal}(K/\mathbb{Q}) \arrow[u, "\cong"']
\end{tikzcd}$$

$$\begin{array}{ccc}
\mathbb{Q}(\zeta_{|\Delta_K|}) & & \\
\cup\!\!\mid & & \\
K & \vdots & \mathbb{Q}(\zeta_p) \\
& & \cup\!\!\mid \\
& \vdots & \mathbb{Q}(\sqrt{p^*})
\end{array}$$

Recall:

$$S_{ram} = \{\, p \mid \Delta_K \,\}$$

$$S_{inert} = \{\, p \nmid \Delta_K,\ \chi_K(p) = -1 \,\}$$

$$S_{split} = \{\, p \nmid \Delta_K,\ \chi_K(p) = 1 \,\}$$

More generally, a Dirichlet char. is a hom.

$$\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$$

$$\Rightarrow \quad L(\chi, s) := \prod_{p \text{ prime}} \frac{1}{1-\chi(p)\cdot p^{-s}}$$

$$= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

use $\chi(n\cdot m) = \chi(n)\cdot\chi(m)$

Here $\chi(n) = 0$ if $\gcd(n, N) \neq 1$

Usually, assumes χ primitive, i.e. χ does not factor

over
$$\begin{tikzcd} (\mathbb{Z}/N)^\times \arrow[r, "\chi"] \arrow[dr] & \mathbb{C}^\times \\ & (\mathbb{Z}/d\mathbb{Z})^\times \arrow[u, dashed] \end{tikzcd}, \quad d \mid N, \; d \neq N$$

If $K/\mathbb{Q}$ quadr., $\chi_K : (\mathbb{Z}/\Delta_K)^\times \to \mathbb{C}^\times$ primitive

$$\zeta_K(s) = \zeta(s) \cdot L(\chi_K, s)$$

More generally, $K \subseteq \mathbb{Q}(\zeta_N)$,

$G := \mathrm{Gal}(K/\mathbb{Q})$

$$\Rightarrow \zeta_K(s) = \prod_{\chi \in \widehat{G}} L(\chi, s)$$

more prec. ass. prim. χ

$\widehat{H} = \mathrm{Hom}(H, \mathbb{C}^\times)$, H finite abelian grp

you

Note: $\widehat{G} \leq \widehat{(\mathbb{Z}/N)^\times}$

$$L(\chi_1, s) = \zeta(s), \quad \chi_1 : (\mathbb{Z}/1)^\times \to \mathbb{C}^\times$$

$$\begin{tikzcd} (\mathbb{Z}/N)^\times \arrow[r, two heads] \arrow[dr] & G \arrow[d, "\chi"] \\ & \mathbb{C}^\times \end{tikzcd}$$

$$\rightsquigarrow \hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$$

$$\big\downarrow \text{ pull pack}$$

$$\mathrm{Hom}((\mathbb{Z}/N)^\times, \mathbb{C}^\times)$$

$$\begin{array}{c} \| \\ \widehat{(\mathbb{Z}/N)^\times} \end{array} \quad)$$

La: $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$

1) $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ conv.

abs. for $\mathrm{Re}\, s > 1$

2) For $\mathrm{Re}\, s > 1$, $L(\chi, s) = \prod_p \frac{1}{1 - \chi(p) p^{-s}}$

3) If $\chi \neq 1 \Rightarrow L(\chi, s)$ has

⚠ cont. into hol. fct. for $\operatorname{Re} s > 0$

($\Rightarrow$ can try to evaluate $L(\chi, s)$ at $s = 1$!)

Pf: 1) $\left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma)$

2) Follows from $\chi(n \cdot m) = \chi(n)\chi(m)$

3) Set $S_t := \sum_{n \leq t} \chi(n), \; t \geq 1$

STP $S_t = O(1)$ (last time)

This follows from La ($H = (\mathbb{Z}/N)^*$)

$S_1 = \chi(1)$, $S_2 = \chi(1) + \chi(2)$

$S_N = \sum_{n \leq N} \chi(n) = \sum_{n \leq N, (n,N)=1} \chi(n) = 0 \ldots$

La: H finite, ab., $\chi : H \to \mathbb{C}^\times$
non-trivial

Then $$\sum_{h \in H} \chi(h) = 0$$

Prf: Pick $g \in H$ with $\chi(g) \neq 1$

$$\Rightarrow \chi(g) \sum_{h\in H} \chi(h) = \sum_{h \in H} \underbrace{\chi(g)\cdot\chi(h)}_{\chi(g\cdot h)}$$

$$\underset{\text{reindex}}{=} \sum_{h\in H} \chi(h) \Rightarrow \text{claim} \quad \square$$

Corollary: $K \subseteq \mathbb{Q}(\zeta_N)$, $G = \mathrm{Gal}(K/\mathbb{Q})$

$$\Rightarrow \prod_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} L(\chi, \cancel{s}\,1) = \frac{2^{r_1}(2\pi)^{r_2} R_K \cdot h_K}{w \cdot \sqrt{|\Delta_K|}}$$

to ass. prim. repr.

Fix $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ n.t., primitive

Want to evaluate $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

at $s = 1$, i.e.

$$\text{"} L(\chi, 1) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \text{"}$$

Observation:

1) $\sum_{n=1}^{\infty} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$ for all

$x \in \mathbb{C}$, s.t. $x \neq 1$

& $|x| \leq 1$

e.g. $x = \zeta$ non-triv root of unity

2) ζ = prim. N-th root of unity

2) $$\chi(n)\cdot \underbrace{\sum_{a\in(\mathbb{Z}/N)} \bar{\chi}(a)\cdot \zeta^a}_{\tau(\bar{\chi}) \text{ "Gauss sum associated to } \bar{\chi}\text{"}} = (*)$$

$$(*) = \sum_{a\in \mathbb{Z}/N} \chi(n\cdot a^{-1})\cdot \zeta^a$$

$$= \sum_{b\in(\mathbb{Z}/N)^\times} \chi(b^{-1})\cdot(\zeta^b)^n = \sum_{b\in(\mathbb{Z}/N)} \bar{\chi}(b)(\zeta^b)^n$$

$b^{-1} = n a^{-1}$

$$\Rightarrow L(\chi,s)\cdot \tau(\bar{\chi}) = \sum_{b\in(\mathbb{Z}/N)} \bar{\chi}(b)\cdot \underbrace{\sum_{n=1}^{\infty}\frac{(\zeta^b)^n}{n^s}}_{\substack{\to -\log(1-\zeta^b) \\ s\searrow 1}}$$

$b\neq 1)$

If $\tau(\bar{\chi}) \neq 0$ (satisfied by prim. of χ)

$$\Rightarrow \quad L(\chi, 1) = \frac{1}{\tau(\bar{\chi})} \cdot \sum_{b \in \mathbb{Z}/N} \bar{\chi}(b) \cdot \log(1-\zeta^b)$$

Now:
$$\log(1-\zeta^b) = \log|1-\zeta^b| + \pi i \left(\frac{a}{N} - \frac{1}{2}\right)$$

(Exercise, Hint: More gen.
$$1 - e^{i\theta} = \underset{\substack{\uparrow \\ \mathbb{R}}}{R} \cdot e^{\frac{i}{2}(\theta - \pi)}$$
)

Let's make some simplifying assumptions (gen. case in Tian)

(12)

$$\chi = \chi_K, \quad K = \mathbb{Q}(\sqrt{-p}), \quad p \equiv 3 (4)$$

$$\mathbb{Q}(\sqrt{-p}) \subseteq \mathbb{Q}(\zeta_p) \qquad p \geq 7$$

$(\leadsto \chi = \overline{\chi}$, ~~[illegible]~~ primitive $\chi : (\mathbb{Z}/p)^* \to \{\pm 1\} \subseteq \mathbb{C}^*$,

$$\Delta_K = -p, \quad \chi(-1) = -1, \quad w = 2)$$

Then: $\dfrac{2\pi \cdot h}{w \cdot \sqrt{p}} = L(\chi, 1)$

$$= \frac{1}{\tau(\chi)} \sum_{1 \leq a < p} \chi(a) \cdot \log(1 - \zeta^a)$$

$$= \frac{1}{\tau(\chi)} \sum_{a=1}^{p-1} \chi(a)\, \pi i \frac{a}{p} + \frac{1}{\tau(\chi)} \underbrace{\sum_{a=1}^{p-1} \chi(a) \cdot \left(-\frac{\pi i}{2}\right)}_{=0,\ \text{by la}}$$

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$$\frac{1}{\tau(\chi)}\left(+ i\sum_{a=1}^{p-1} \chi(a)\left(\log|1-\zeta^a|\right)\right)$$

$$= \frac{1}{2}\sum_{a=1}^{p-1}\underbrace{(\chi(a)+\chi(-a))}_{=0,\text{ as } \chi(-1)=-1}\cdot\log|1-\zeta^a|$$

$$\log|1-\zeta^a| = \log|1-\zeta^{-a}|$$

$$= \frac{1}{\tau(\chi)}\sum_{a=1}^{p-1} \chi(a)\cdot \pi i\cdot \frac{a}{p}$$

La: $\tau(\chi)\cdot\tau(\chi) = -p$

$(\Rightarrow \tau(\chi) = \pm i\sqrt{p}$, actually $\tau(\chi) = i\sqrt{p}\,)$

Proof: $\tau(\chi)\cdot\tau(\chi)$

$$= \sum_{a,b\in(\mathbb{Z}/p)} \chi(a)\cdot\chi(b)\cdot\zeta^{a+b}$$

$$= \sum_{a,c \in (\mathbb{Z}/p)^\times} \chi(a) \cdot \chi(a \cdot c) \cdot \zeta^{a(1+c)}$$ 14

$b = ac$

$$= \sum_{c \in (\mathbb{Z}/p)^\times} \chi(c) \underbrace{\sum_{a \in (\mathbb{Z}/p)^\times} \zeta^{a(1+c)}}_{= \begin{cases} -1, & c \neq -1 \\ p-1, & c = -1 \end{cases}} \qquad = (\zeta^{1+c})^a$$

$\chi^2 = 1$

$$\left(\frac{X^p - 1}{X - 1} = 1 + X + \dots + X^{p-1} \right)$$

$$= p \cdot \chi(-1) \underbrace{- \sum_{c \in (\mathbb{Z}/p)^\times} \chi(c)}_{= 0, \text{ byla}}$$

$$= -p$$

$\square$

Get $\frac{\cancel{2}\pi \cdot h}{\cancel{w} \cdot \sqrt{p}} = L(\chi, 1)$

$$= \pm \frac{1}{i\sqrt{p}} \sum_{a=1}^{p-1} \chi(a) \cdot \pi i \cdot \frac{a}{p}$$

$$\Leftrightarrow \quad h = \pm \frac{1}{p} \sum_{a=1}^{p-1} \chi(a) \cdot a$$

$$= \mp \frac{1}{p} \left(\sum_{\substack{a=1 \\ \chi(a)=-1}}^{p-1} a - \sum_{\substack{a=1 \\ \chi(a)=1}}^{p-1} a \right)$$

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Set $N := \{1 \leq a < p \mid \chi(a) = -1\}$

$= \{1 \leq a < p \mid a$ not a quadr. residue mod $p\}$

$R := \{1 \leq a < p \mid \chi(a) = 1\}$

$= \{1 \leq a < p \mid a$ is a quadr. res. mod $p\}$

Namely,

$$\chi(a) = \left(\frac{a}{p}\right)$$

In the end,

$$h = \mp \frac{1}{p}\left(\sum_{a \in N} a - \sum_{a \in R} a\right)$$

$$= \left(\frac{p-1}{2} - \frac{2}{p}\sum_{a \in R} a\right)$$

Ex: 1) $K = \mathbb{Q}(\sqrt{-7})$

$\Rightarrow R = \{1, 2, 4\}$

$\Rightarrow h = \frac{7-1}{2} - \frac{2}{7}(1+2+4)$

$= 3 - 2 = 1$

$$\begin{array}{ccc} & (-)^2 & \\ 1 & \mapsto & 1 \\ 2 & & 4 \\ 3 & & 2 \\ & & \bmod 7 \end{array}$$

2) $K = \mathbb{Q}(\sqrt{-11})$

$\Rightarrow R = \{1, 3, 4, 5, 9\}$

$\Rightarrow h = 5 - \frac{2}{11}(1+3+4+5+9)$

$= 5 - 4 = 1$

$$\begin{array}{ccc} & (-)^2 & \\ 1 & \mapsto & 1 \\ 2 & & 4 \\ 3 & & 9 \\ 4 & & 5 \\ 5 & & 3 \\ & & \bmod 11 \end{array}$$

3) $K = \mathbb{Q}(\sqrt{-19})$

$\Rightarrow h = 1$

4) $K = \mathbb{Q}(\sqrt{-23})$

Start to use $\left(\frac{p}{23}\right)$

$$\left(\frac{2}{23}\right) = (-1)^{\frac{23^2-1}{8}} = 1$$

$$\left(\frac{3}{23}\right) = \left(\frac{-23}{3}\right) = \left(\frac{1}{3}\right) = 1$$

$5 \quad \ldots \quad = -1$

$7 \quad = -1$

$11 \quad = -1$

$13 \quad = 1$

$17 \quad = -1$

$\frac{19}{23} \quad = -1$

$\Rightarrow R = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$

$\Rightarrow h = 11 - \frac{2}{23}(92) = 11 - 8 = 3$ (☺)